Partitions with parts in a finite set *

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Abstract

Let A be a nonempty finite set of relatively prime positive integers, and let $p_A(n)$ denote the number of partitions of n with parts in A. An elementary arithmetic argument is used to prove the asymptotic formula

$$p_A(n) = \left(\frac{1}{\prod_{a\in A} a}\right)\frac{n^{k-1}}{(k-1)!} + O\left(n^{k-2}\right).$$

Let A be a nonempty set of positive integers. A *partition* of a positive integer n with parts in A is a representation of n as a sum of not necessarily distinct elements of A. Two partitions are considered the same if they differ only in the order of their summands. The *partition function* of the set A, denoted $p_A(n)$, counts the number of partitions of n with parts in A.

If A is a finite set of positive integers with no common factor greater than 1, then every sufficiently large integer can be written as a sum of elements of A (see Nathanson [3] and Han, Kirfel, and Nathanson [2]), and so $p_A(n) \geq 1$ for all $n \geq n_0$. In the special case that A is the set of the first k integers, it is known that

$$p_A(n) \sim \frac{n^{k-1}}{k!(k-1)!}.$$

Erdős and Lehner[1] proved that this asymptotic formula holds uniformly for $k = o(n^{1/3})$. If A is an arbitrary finite set of relatively prime positive integers, then

$$p_A(n) \sim \left(\frac{1}{\prod_{a\in A} a}\right)\frac{n^{k-1}}{(k-1)!}. \tag{1}$$

The usual proof of this result (Netto [4], Pólya–Szegö [5, Problem 27]) is based on the partial fraction decomposition of the generating function for $p_A(n)$. The purpose of this note is to give a simple, purely arithmetic proof of (1).

We define $p_A(0) = 1$.

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Theorem 1 *Let $A = \{a_1, \ldots, a_k\}$ be a set of k relatively prime positive integers, that is,*

$$\gcd(A) = (a_1, \ldots, a_k) = 1.$$

Let $p_A(n)$ denote the number of partitions of n into parts belonging to A. Then

$$p_A(n) = \left(\frac{1}{\prod_{a \in A} a} \right) \frac{n^{k-1}}{(k-1)!} + O\left(n^{k-2}\right).$$

Proof. Let $k = |A|$. The proof is by induction on k. If $k = 1$, then $A = \{1\}$ and

$$p_A(n) = 1,$$

since every positive integer has a unique partition into a sum of 1's.

Let $k \geq 2$, and assume that the Theorem holds for $k - 1$. Let

$$d = (a_1, \ldots, a_{k-1}).$$

Then

$$(d, a_k) = 1.$$

For $i = 1, \ldots, k-1$, we set

$$a_i' = \frac{a_i}{d}.$$

Then

$$A' = \{a_1', \ldots, a_{k-1}'\}$$

is a set of $k - 1$ relatively prime positive integers, that is,

$$\gcd(A') = 1.$$

Since the induction assumption holds for A', we have

$$p_{A'}(n) = \left(\frac{1}{\prod_{i=1}^{k-1} a_i'} \right) \frac{n^{k-2}}{(k-2)!} + O\left(n^{k-3}\right)$$

for all nonnegative integers n.

Let $n \geq (d-1)a_k$. Since $(d, a_k) = 1$, there exists a unique integer u such that $0 \leq u \leq d - 1$ and

$$n \equiv u a_k \pmod{d}.$$

Then

$$m = \frac{n - u a_k}{d}$$

is a nonnegative integer, and

$$m = O(n).$$

If v is any nonnegative integer such that

$$n \equiv v a_k \pmod{d},$$

then $va_k \equiv ua_k \pmod{d}$ and so $v \equiv u \pmod{d}$, that is, $v = u + \ell d$ for some nonnegative integer ℓ. If

$$n - va_k = n - (u + \ell d)a_k \geq 0,$$

then

$$0 \leq \ell \leq \left[\frac{n}{da_k} - \frac{u}{d}\right] = \left[\frac{m}{a_k}\right] = r.$$

We note that

$$r = O(n).$$

Let π be a partition of n into parts belonging to A. If π contains exactly v parts equal to a_k, then $n - va_k \geq 0$ and $n - va_k \equiv 0 \pmod{d}$, since $n - va_k$ is a sum of elements in $\{a_1, \ldots, a_{k-1}\}$, and each of the elements in this set is divisible by d. Therefore, $v = u + \ell d$, where $0 \leq \ell \leq r$. Consequently, we can divide the partitions of n with parts in A into $r + 1$ classes, where, for each $\ell = 0, 1, \ldots, r$, a partition belongs to class ℓ if it contain exactly $u + \ell d$ parts equal to a_k. The number of partitions of n with exactly $u + \ell d$ parts equal to a_k is exactly the number of partitions of $n - (u + \ell d)a_k$ into parts belonging to the set $\{a_1, \ldots, a_{k-1}\}$, or, equivalently, the number of partitions of

$$\frac{n - (u + \ell d)a_k}{d}$$

into parts belonging to A', which is exactly

$$p_{A'}\left(\frac{n - (u + \ell d)a_k}{d}\right) = p_{A'}\left(m - \ell a_k\right).$$

Therefore,

$$\begin{aligned}
p_A(n) &= \sum_{\ell=0}^{r} p_{A'}(m - \ell a_k) \\
&= \left(\frac{1}{\prod_{i=1}^{k-1} a'_i}\right) \sum_{\ell=0}^{r} \left(\frac{(m - \ell a_k)^{k-2}}{(k-2)!} + O(m^{k-3})\right) \\
&= \left(\frac{d^{k-1}}{\prod_{i=1}^{k-1} a_i}\right) \sum_{\ell=0}^{r} \frac{(m - \ell a_k)^{k-2}}{(k-2)!} + O(n^{k-2}).
\end{aligned}$$

To evaluate the inner sum, we note that

$$\sum_{\ell=0}^{r} \ell^j = \frac{r^{j+1}}{(j+1)} + O(r^j)$$

and

$$\sum_{j=0}^{k-2} (-1)^j \binom{k-1}{j+1} = -\sum_{j=1}^{k-1} (-1)^j \binom{k-1}{j} = 1.$$

Then

$$
\begin{aligned}
\sum_{\ell=0}^{r} \frac{(m-\ell a_k)^{k-2}}{(k-2)!} &= \frac{1}{(k-2)!} \sum_{\ell=0}^{r} \sum_{j=0}^{k-2} \binom{k-2}{j} m^{k-2-j} (-\ell a_k)^j \\
&= \frac{1}{(k-2)!} \sum_{j=0}^{k-2} \binom{k-2}{j} m^{k-2-j} (-a_k)^j \sum_{\ell=0}^{r} \ell^j \\
&= \frac{1}{(k-2)!} \sum_{j=0}^{k-2} \binom{k-2}{j} m^{k-2-j} (-a_k)^j \left(\frac{r^{j+1}}{(j+1)} + O(r^j) \right) \\
&= \frac{1}{(k-2)!} \sum_{j=0}^{k-2} \binom{k-2}{j} m^{k-2-j} (-a_k)^j \left(\frac{m^{j+1}}{a_k^{j+1}(j+1)} + O(m^j) \right) \\
&= \frac{m^{k-1}}{a_k} \sum_{j=0}^{k-2} \binom{k-2}{j} \frac{(-1)^j}{(k-2)!(j+1)} + O(m^{k-2}) \\
&= \frac{m^{k-1}}{a_k} \sum_{j=0}^{k-2} \frac{(-1)^j}{(k-2-j)!j!(j+1)} + O(m^{k-2}) \\
&= \frac{m^{k-1}}{a_k} \sum_{j=0}^{k-2} \frac{(-1)^j}{(k-1-(j+1))!(j+1)!} + O(m^{k-2}) \\
&= \frac{m^{k-1}}{a_k(k-1)!} \sum_{j=0}^{k-2} (-1)^j \binom{k-1}{j+1} + O(m^{k-2}) \\
&= \frac{m^{k-1}}{a_k(k-1)!} + O(m^{k-2}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
p_A(n) &= \left(\frac{d^{k-1}}{\prod_{i=1}^{k-1} a_i} \right) \sum_{\ell=0}^{r} \frac{(m-\ell a_k)^{k-2}}{(k-2)!} + O(n^{k-2}) \\
&= \left(\frac{d^{k-1}}{\prod_{i=1}^{k-1} a_i} \right) \left(\frac{m^{k-1}}{a_k(k-1)!} + O(n^{k-2}) \right) + O(n^{k-2}) \\
&= \left(\frac{d^{k-1}}{\prod_{i=1}^{k-1} a_i} \right) \left(\frac{1}{a_k(k-1)!} \right) \left(\frac{n}{d} - \frac{u a_k}{d} \right)^{k-1} + O(n^{k-2}) \\
&= \left(\frac{d^{k-1}}{\prod_{i=1}^{k-1} a_i} \right) \left(\frac{1}{a_k(k-1)!} \right) \left(\frac{n}{d} \right)^{k-1} + O(n^{k-2}) \\
&= \left(\frac{1}{\prod_{i=1}^{k} a_i} \right) \frac{n^{k-1}}{(k-1)!} + O(n^{k-2}).
\end{aligned}
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This completes the proof.

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